SOLUTIONS TO PRISM PROBLEMS

2019

<u>NOTE:</u> Teachers are strongly encouraged to help students with those solutions below in which they find the steps not obvious. Indeed, studying the solutions also has the advantage of guiding students towards alternative methods, and using their creativity and experience, they will increase their innate ability so they can better tackle Olympiad-type problems.

1. (B) [ANSWER 32]

The sequence $1, 2, 4, 8, 16, \ldots$ follows a geometric pattern (that is, each number after the first, $a,$ is the sample multiple, $r,$ of the previous number. Mathematically, the sequence is of the form $a, ar, ar^2, ar^3, \ldots.$ Here $a = 2$ and $r = 3,$ and the next number in in the given sequence is $2 \times 16 = 32.$

Students will encounter very useful results about geometric (and many other types of) sequences in their future education!

2. (A) [ANSWER 12]

Since SJ first gave a fraction $\frac{2}{3}$ of her 12 weets to SS, SJ gave $\frac{2}{3} \times 12 = 8$ sweeties to SS. SJ now has $12 - 8 = 4$ sweets remaining and gave one-quarter of these , i.e. 1 to SS. SJ now has $12 - 8 - 1 = 3$ sweeties and SS has $24 - 3 = 21$ sweeties. Since finally SS gave 9 sweeties to SJ, SJ will now have $3 + 9 = 12$

sweets and so SS will have only $24 - 12 = 12$ sweets. In conclusion, the twins will each have the same number of sweets (12) as each had originally .

3. (E) [Answer 110]

Since $x - y = 10,$ we have (on squaring both sides) , $x^2 + y^2 - 2xy = 100.$ But $xy = 5$

(given) , so we have $x^2 + y^2 = 1 = +2(5) = 110.$

4. (D) [Answer 26]

Let x be Peter's age today and let y be Charlie's age today. 10 years ago, Peter was then aged $x - 10$ and Charlie's age was $y - 10,$ and we are told that

$$x - 10 = 2(x - 10) \qquad \textbf{(*)}$$

Fourteen years ago, Peter's age was $x - 14$ and we are given that this is three times what Charlie's age was then, that is,

$$x - 14 = 3(y - 14) \qquad \textbf{(**)}$$

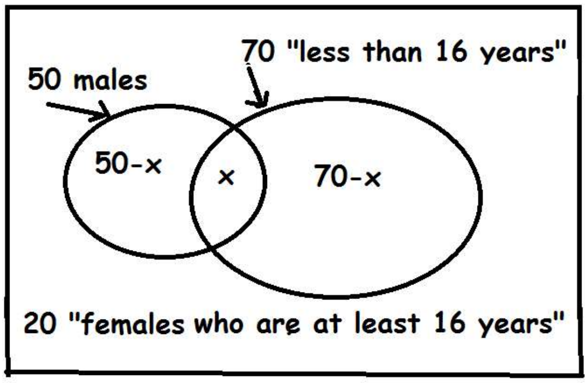
We now have two simultaneous equations (*) and (**) to solve for x (and y if we wish).

We re-write (*) as $x - 2y = 10$ and we re-write (**) as $x - 3y = -28.$

Solving these two equations for x and y gives $x = 26$ and $y = 18.$ In particular, Peter's current age is 26.

5. (D) [Answer 40]

100 students



In the Venn diagram above, the entire rectangle represents all 100 students. The smaller 'egg' represents the set of 50 male students and the larger egg is the set of 70 students who are less than 16 years of age.

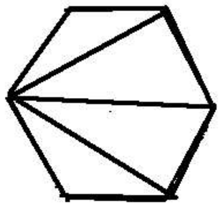
If we let x be the number of students who are male and less than 16 years of age, these x students form the intersection of the two 'eggs'.

Notice that the 20 students who are females and are at least 16 years of age are represented by the portion of the rectangle that is outside of both 'eggs'.

The total number of students, 100, is the sum of the numbers $50-x, x, 70-x$ and 20, where $50-x$ is the number of students who are male and at least 16 years old and $70-x$ is the number of students who are female and less than 16 years of age. That is, $100 = 50-x+x+70-x+20$. Hence $100 = 140-x$, so $x = 40$ is the required answer.

6. (A) [120 degrees]

One way of proceeding is to connect vertices of the regular hexagon to form 4 triangles as shown below.

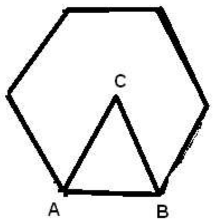


Clearly the sum of all 6 angles of the hexagon equals the sum of the angles in all 4 triangles. But the sum of the angles in the 4 triangles is $4 \times 180° = 720°$.

Thus the measure of the angle between any two adjacent sides of the hexagon is $\frac{720°}{6} = 120°$.

Another way:

Alternatively, the diagram below (not drawn very accurately!) shows one of the 6 triangles that can be formed by joining the centre of the hexagon to the 6 vertices.



The measure of the angle $\angle ACB$ is $\frac{360°}{6} = 60°$ and since the three angles of the triangle sum to $180°$ and the triangle is isosceles, the measure of the angle $\angle BAC$ is $\frac{1}{2}(180° - 60°) = 60°$. Hence the measure of the angle between any two adjacent sides is $2 \times 60° = 120°$,. agreeing with the answer in our earlier method above.

7. (E) [Answer $(27)^{\frac{2}{3}} < 8^{\frac{4}{3}} < 9^{\frac{3}{2}}$]

We work out the values of each of the numbers $8^{\frac{4}{3}}$, $9^{\frac{3}{2}}$ and $(27)^{\frac{2}{3}}$, and then simply rank them from smallest to largest.

Recalling that one of the Laws of indices says that $x^{ab} = (x^a)^b$. Hence $8^{\frac{4}{3}} = (8^{\frac{1}{3}})^4 = 2^4 = 16$. (Note here that $8^{\frac{1}{3}} = \sqrt[3]{8}$, i.e. the cubed root of 8, and this is 2 because $2 \times 2 \times 2 = 8$.)

Next, $9^{\frac{3}{2}} = (9^{\frac{1}{2}})^3 = 3^3 = 27$.

Finally, $(27)^{\frac{2}{3}} = ((27)^{\frac{1}{3}})^2 = 3^2 = 9$.

Since $9 < 16 < 27$, the relationship between $8^{\frac{4}{3}}$, $9^{\frac{3}{2}}$ and $(27)^{\frac{2}{3}}$ is then $(27)^{\frac{2}{3}} < 8^{\frac{4}{3}} < 9^{\frac{3}{2}}$

8. (C) [Answer 95 metres]

The ratio of Bridget's speed to Alice's speed is $\frac{450}{500} = 0.9$ because when Alice crosses the finish line, Bridget will have covered 450 metres. Thus Bridget is travelling at 90% of Alice's speed. Similarly, Connie travels at 90% of Bridget's speed so Connie travels at $(90\%$ of $90\%) = 81\%$ of Alice's speed. Accordingly, when Alice has covered the 500 metres, Connie will have covered $0.81 \times 500 = 405$ metres. Hence Alice beat Connie by $500 - 405 = 95$ metres.

9. (A) [ANSWER: -3a]

Note that many students will solve this problem rapidly by drawing a number line and positioning each of the five given choices on this line to easily see that $-3a$ is the largest among them.

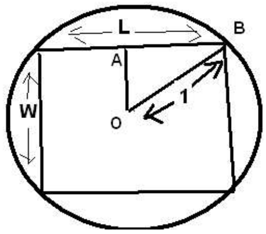
If instead we proceed in a formal manner, we can algebraically compare $-3a$ with each of the other possible answers in turn. Notice first that $-3a > 6a$ because this inequality is the same as $-9a > 0$ which is true if (dividing across by -9) $a < 0$, and since indeed $a < 0$ (given) the number $-3a$ indeed exceeds the number $6a$. [Here and below it is important for students to note that when dealing with inequalities, we change the sign of the inequality when we multiply or divide across by a negative number.]

Next $-3a$ is greater than $-a + 1$ because the inequality $-3a > -a + 1$ is the same as $-2a > 1$ which is equivalent to $a < -\frac{1}{2}$, and this is true since we are given that a is a negative integer.

Next, observe that $-3a$ is greater than $-a$ because the inequality $-3\dot{a} > -a$ is the same as $-2a > 0$ or $a < 0$, which is given.

Finally, we have $-3a > -a - 2$ because this inequality is equivalent to $-2a > -2$ which is true if and only if $a < \frac{-2}{-2} = 1$, and this is true since we are given that a is negative

10. (C) {ANSWER 2]



The figure above shows a rectangle with circumscribed circle. In the right-angled triangle OAB, the length$|OB|$ is the radius, 1, of the circle, the length $|OA|$ is one-half of the width W of the rectangle and the length $|AB|$ is one-half of the length L of the rectangle. By Pythagoras" Theorem, $\left(\frac{W}{2}\right)^2 + \left(\frac{L}{2}\right)^2 = 1^2$, so we have $W^2 + L^2 = 4$.

Our first goal is to find the values for W and L that maximize the area WL of the rectangle, subject to the condition that $W^2 + L^2 = 4$.

At this stage some students might use enlightened guesswork, realizing that the largest rectangle will (by symmetry) be a square. If this logic is valid then we'd put $W = L$, so from $W^2 + L^2 = 4$ we'd have $2W^2 = 4$, then $W^2 = 2$ and $W = \sqrt{2}$. The largest rectangle would then have area $WL = WW = 2$.

This is actually the correct answer and we'd expect most students not to have time to make a derivation that does not capitalize on the symmetry in the problem.

Longer (calculus) derivation: (Note that knowledge of calculus is not required in this or Maths Olympiad examinations.)

If one does not use symmetry, then another approach is to use calculus. From $W^2 + L^2 = 4$, we have $W = \sqrt{L^2 - 4} = (L^2 - 4)^{1/2}$. The area of the rectangle is Area $= LW = L(L^2 - 4)^{1/2}$. We will find the value for L that maximizes this area. Differentiating the function Area $= L(L^2 - 4)^{1/2}$ with respect to L gives (using the chain and product rules)

$$\frac{d(\text{Area})}{dL} = L \times \frac{1}{2} \times 2L \times (L^2 - 4)^{-1/2} + 1 \times (L^2 - 4)^{1/2}.$$

Setting this equal to 0 for an extremal point, we have

$$L \times \frac{1}{2} \times 2L \times (L^2 - 4)^{-1/2} + 1 \times (L^2 - 4)^{1/2} = 0.$$

Dividing both sides by $(L^2 - 4)^{-1/2}$, we get

$L^2 + (L^2 - 4) = 0.$ Hence $L^2 = 2$ and $L = \sqrt{2}$

(It is easy to check that this value $L = \sqrt{2}$ does indeed maximize the function Area $= L(L^2 - 4)^{1/2}$ (interested students can check by examining he second derivative of the function).

Since $L = \sqrt{2}$ and $W^2 + L^2 = 4$ we must have $W = \sqrt{2}$ also.

Hence the maximum area of the rectangle is $LW = \sqrt{2} \times \sqrt{2} = 2$, as we had obtained above by a simpler symmetry argument.

11. (D)

12. (C)

A fraction $\left(\frac{1}{10} + \frac{1}{20} - \frac{1}{18000}\right) = \frac{2699}{18000}$ of the tank will be filled in one minute. (Students: If this is unclear, please consult with your mathematics teacher.) Hence the entire tank will be filled in $\frac{18000}{2699} = 6.669$ minutes, and to the nearest minute, this is 7 minutes.

Note: It may be useful to note that this question involves *approximation* of some quantity (the time for the tank to be filled). In approximating the answer to a problem, it often simplifies calculations greatly if we omit or round quantities that will have only a negligible effect on the answer. In the present problem, alert students may have noticed that since the rate at which the tank is emptied by pump C is very small in comparison with the rates at which A and B fill the tank, one could obtain a very good approximation to the answer by completely omitting pump C from our calculation. In fact, notice that $\frac{1}{10} + \frac{1}{20} - \frac{1}{18000}$ is very close to $\frac{1}{10} + \frac{1}{20}$ and then $\frac{1}{\frac{1}{10} + \frac{1}{20} - \frac{1}{18000}} = \frac{18000}{2699} = 6.669$ is very close to $\frac{1}{\frac{1}{10} + \frac{1}{20}} = \frac{200}{30} = 6.667,$ which is the time it takes for the tank to be filled if only pumps A and B are in operation. Incidentally, students might further note that the quantity $\frac{1}{\frac{1}{10} + \frac{1}{20}}$ is proportional to the harmonic mean of the numbers 10 and 20.

13. (C)

$P(x) = x^3 - 7x + 6.$ It is easy to see that one of the roots is $x = 1.$ Indeed $P(1) = (1)^3 - 7(1) + 6 = 1 - 7 + 6 = 0$

It follows that $x - 1$ is a factor. Dividing $x - 1$ into $x^3 - 7x + 6$ we see that $x^3 - 7x + 6 = (x-1)(x^2 - 2x + 6),$ Factoring $x^2 - 2x + 6,$ we have

$x^2 - 2x + 6 = (x-2)(x+3),$ so the other two roots are $x = 2$ and $x = -3.$ (Some students may have fond these two roots by trial and error, in the

same way as they may have obtained the root $x = 1.$) The sum of the three roots is then $1 + 2 + (-3) = 0.$

14. (D)

Let $x = \sqrt{\frac{1}{2} + \sqrt{\frac{1}{2} + \sqrt{\frac{1}{2} + \sqrt{\frac{1}{2} + \ldots}}}}$.

Squaring both sides gives $x^2 = \left(\sqrt{\frac{1}{2} + \sqrt{\frac{1}{2} + \sqrt{\frac{1}{2} + \sqrt{\frac{1}{2} + \ldots}}}}\right)^2$, that is,

$x^2 = \frac{1}{2} + \sqrt{\frac{1}{2} + \sqrt{\frac{1}{2} + \sqrt{\frac{1}{2} + \ldots}}}$.

But this can be written as

$x^2 = \frac{1}{2} + x.$

We now solve this quadratic equation for $x.$

The equation is $x^2 - x - \frac{1}{2} = 0$. Applying the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ for the roots of a quadratic equation $ax^2 + bx + c = 0$ we find (with $a = 1, b = -1$ and $c = -\frac{1}{2}$)

that $x = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(1)\left(-\frac{1}{2}\right)}}{2(1)}$, i.e. $x = \dfrac{1 \pm \sqrt{3}}{2}$.

But clearly $x = \sqrt{\frac{1}{2} + \sqrt{\frac{1}{2} + \sqrt{\frac{1}{2} + \sqrt{\frac{1}{2} + \ldots}}}}$ is positive, so the solution is $x = \dfrac{1 + \sqrt{3}}{2}$.

Note: Of course students should not feel in any way bad if they did not work out the solution to theis problem. The technique of squaring both sides of

$x = \sqrt{\frac{1}{2} + \sqrt{\frac{1}{2} + \sqrt{\frac{1}{2} + \sqrt{\frac{1}{2} + \ldots}}}}$ as a way of getting an easy quadratic equation to solve for x is a ‘trick’ that is very useful in mathematics but far from obvious.

14. (D) [ANSWER 600,000]

Here is one way of proceeding. Since no 7 appears in the either of the two numbers 000000 or 999999, counting the number of 7s that will appear when all the numbers 1 to one million are written out is the same as counting the number of 7s that will appear when all of the numbers 000000 through to 999999 are written out. All of these one million numbers are six-digit numbers $a_1a_2a_3a_4a_5a_6$, each digit of which can be a 0, a 1, a 2,..., or a 9 (ten choices for each digit).

Now if the singles digit a_6 is to be a 7, any of the other five digits a_1, a_2, a_3, a_4 and a_5 can be any of 0,1,2,... or 9. So there are $10 \times 10 \times 10 \times 10 \times 10 = 100,000$ choices of numbers for which a 7 appears as the singles digit (examples are 656897, 772137, etc.). Similarly. there will be 100000 numbers whose second-last digit a_5 will be a 7, and so on to the number of choices of numbers whose leading digit a_2 is 7. The total number of appearances of the digit 7 will then be

$100,000 + 100,000 + 100,000 + 100,000 + 100,000 + 100,000 = 600,000.$

Note: Some students might argue that if we write out all of the numbers 000000 through to 999999, the total number of digits written will be 6×1 million, i.e. 6,000,000. Since each digit 0,1,2,...,9 will appear the same number of times, a fraction $\frac{1}{10}$ of the 6,000,000 digits will be 7’s, so indeed there will be 600,000 appearances of 7.

15. (A) [ANSWER 0]

We prove by contradiction that there is no prime number p for which $2p+1$ is a perfect square. So suppose p is a prime number and $2p+1$ *is* a perfect square. Now clearly $2p+1$ is odd. We now clam that every odd perfect square a is of the form $a = 4m+1$.

Proof that every odd perfect square a is of the form $a = 4m+1$: First note that we can write *any* integer b in the form $b = 4r + s$ for some integer r and $s = 0, 1, 2$ or 3 (this is possible since if we divide an integer b by 4 we will get either no remainder, a remainder of 1, a remainder of 2 or a remainder of 3).

Then $b^2 = (4r+s)^2 = 16r^2 + 8rs + s^2 = 4(4r^2 + 2rs) + s^2$.

Since $s^2 = 0, 1, 4$ or 9, we can write $b^2 = 4(4r^2+2rs) + s^2$ in the form $b^2 = 4u + v$ where $v = 0$ or 1.

Now if $a = b^2$ is an odd perfect square then we must have $v = 1$. So we have shown that every odd perfect square a is of the form $a = 4m+1$.

(*Side note:* Mathematicians write statements like $a = 4m+1$ as "a is congruent to 1 modulo 4" or, for short, "$a \equiv 1$ (mod) 4"

So now we have that $2p+1$ is of the form $4m+1$. This is equivalent to saying that $2p$ is of the form $4m$, i.e. $2p$ is a multiple of 4. But this means that p is even, so it must be 2 (because 2 is the only even prime number). But this is not possible because we'd then have $2p+1 = 2(2)+1 = 5$ which is not a perfect square.

This contradiction shows that there is no prime number p for which $2p+1$ is a perfect square.

Note:

Of course this question was difficult and we do not expect students to do well in it! However we want to encourage students to study the above solution because it contains a number of nice ideas associated with mathematical proofs and mathematical techniques. Students who plan to study mathematics later at a more abstract level than they currently study in school will encounter plenty of proofs such as that above, and it would be highly beneficial to them if they could now develop an interest in such mathematics! Students are asked to bear in mind also that employers seek employees who can both:

(a) work well as part of a team

and

(b) solve problems.

Mastering solutions to the above type of problem is an excellent way for students to train themselves in the logical thought processes that are essential for problem-solving!
